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EQUATIONS OF PERTURBED MOTION OF AN EQUATORIAL SATELLITE IN THE "ACTION — ANGLE" VARIABLES

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A problem of computing the canonical action — angle variables for an equatorial satellite of an axisymmetric plant is considered, and the Kolmogorov [1] theorem used to study the problem of preserving the conditionally periodic motions of the satellite using a recent form of the equation of perturbed motion.

Let us introduce a rectangular coordinate system centered on the planet, the xy-plane of which coincides with the equatorial plane of the planet. If the planet has axial symmetry about the polar axis, then the force function is determined by the formula [2]

$$U = \frac{fM}{r}\left[1 - \sum_{k=2}^{\infty} \frac{J_k R^k}{r^k} P_k(\sin\varphi)\right] \tag{1(1)}$$

$$\sin\varphi = z / r, \quad r^2 = x^2 + y^2 + z^2$$

where M is the Earth's mass and R its radius.

Introducing the polar r, θ-coordinates in the equatorial plane, we can write the force function for the perturbed problem in the form

$$V = V(r) + \varepsilon F(r, \theta, \varepsilon) \tag{2}$$

where ε is a small parameter. Putting $\varepsilon = 0$, we obtain the unperturbed motion. Retaining in (2) the second zonal harmonic only, we obtain

$$V = \frac{fM}{r} - \frac{fMJ_2R^2}{2r^3}$$

Let us consider the Hamilton — Jacobi equation for the case of an artificial satellite moving in the equatorial plane of the axisymmetric planet. Since θ is a cyclic coordinate we have for the conjugated impulse $\partial W / \partial\theta = \alpha_2 = \text{const}$, and the complete integral of the Hamilton — Jacobi equation

$$W = \int A(r)\, dr + \alpha_2\theta$$

$$A(r) = \left(\frac{fMJ_2R^2}{r^3} - \frac{\alpha_2^2}{r^2} - \frac{2fM}{r} + 2\alpha_1\right)^{1/2}$$

where we restrict ourselves to the case of $\alpha_1 < 0$ (α_1 is the constant of the energy integral).

Next, we pass from the canonical elements α_1, α_2 to other canonical impulses, integrating over the period of variation of r and θ. This yields

$$I_1 = \oint A(r)\, dr, \quad I_2 = \oint \alpha_2 d\theta$$

The quantities I_1 and I_2 are the action variables and the coordinate θ will vary, during the period of motion (when $\alpha_2 \neq 0$) over the range $0 \leqslant \theta \leqslant 2\pi$. The quantity I_1 can be rewritten in the form

$$I_1 = \int_{m_1}^{m_2} [P(u)]^{1/2} du \tag{3}$$

$$P(u) = bu^3 - \alpha_2 u^2 - cu + 2\alpha_1 = 0 \tag{4}$$
$$(u = 1 / r, \ c = 2fM, \ b = fMJ_2R^2)$$

where the limits of integration are positive roots of (4).

Let us investigate the motion of the satellite in the case when the initial values satisfy the conditions

$$|u_0| < \sqrt{c / b}, \quad \dot{\theta} \neq 0, \quad \alpha_2 \neq 0 \quad (u_0 = 1 / r_0)$$

Let us assign to the argument u of the polynomial $P(u)$ the values $-\sqrt{c / b}$, u_0, $+\sqrt{c / b}$, $+\infty$. It is clear that

$$P\left(-\sqrt{\frac{c}{b}}\right) = -\frac{c\alpha_2^2}{b} + 2\alpha_1 < 0, \quad P(u_0) = u_0'^2 \geqslant 0$$
$$P\left(+\sqrt{\frac{c}{b}}\right) = -\frac{c\alpha_2^2}{b} + 2\alpha_1 < 0, \quad P(+\infty) > 0$$

From this it follows that all three roots of the polynomial $P(u)$ are real. One of the roots, namely u_3, is always real and greater than $\sqrt{c / b}$. The remaining two roots u_1 and u_2 lie in the interval $(-\sqrt{c / b}, +\sqrt{c / b})$ and we have $-\sqrt{c / b} < u_1 \leqslant u_0 \leqslant u_2 < \sqrt{c / b} < u_3$.
When $\alpha_1 < 0$, the equation (4) has either one, or three real roots.

The case of three real roots $0 < u_1 < u_2 < u_3$ corresponds to the actual initial velocities imparted to the planet's satellite, and in the case of real motion we have $P(u) \geqslant 0$. The coordinates u assume the values which lie within the interval $u_1 \leqslant u \leqslant u_2$, or within $u_3 \leqslant u < \infty$, and the first case is of practical interest.

The integral (3) can be expressed in terms of the complete elliptic integrals of the first, second and third kind. Denoting by u_0 the smallest root of the polynomial $P(u)$ and $Q = 4abm$, $B = abu_0 - \alpha_2$, $G = 4a\alpha_1$, $N = 4ac$, we find

$$I_1 = \frac{4\eta_2}{\sqrt{ab}} \left[\frac{QK(k) - QE(k)}{k^2} + BK(k) + \left(\frac{G}{u_0^2} - \frac{N}{u_0}\right) \Pi(k, -n)\right] \tag{5}$$

where $K(k)$, $E(k)$ and $\Pi(k, -n)$ are the complete elliptic integrals of the first, second the third kind respectively. Their modulus and parameter are given by the formulas $k = \xi_1 / \xi_2$ and $n = m / u_0$ where ξ_1 and ξ_2 are roots of the corresponding fourth degree equation.

Now we can solve the equation (5) for the initial element α_1 of the orbit, obtaining the latter in the form of a power series in k^2. For $k = 0$ we have the approximate expression

$$\alpha_1 = (I_1 - LI_2 - S)\Phi^{-1}$$
$$L = \frac{\eta_2^2}{2\sqrt{ab}}, \quad \Phi = \frac{2\eta_2\pi}{\sqrt{ab}}(1 - \eta_2^2 u_0)$$

$$S = \frac{4\eta_2}{\sqrt{ab}}\left(\frac{\pi p}{4} + \frac{1}{2}ab\pi u_0 - \frac{N\pi}{2u_0}\right)$$

Let us consider the transformation $(\alpha_1, \alpha_2) \to (I_1, I_2)$ achieved by means of the relation (5) and the formula $I_2 = 2\pi\alpha_2$. The transformation will be single-valued, since

$$\frac{\partial (I_1, I_2)}{\partial (\alpha_1, \alpha_2)} = 2\pi \frac{\partial I_1}{\partial \alpha_1} \neq 0$$

for $\alpha_1 \neq 0$ and $u_0 \neq u_1$. The functions $\alpha_1 = \alpha_1 (I_1, I_2)$, $\alpha_2 = I_2 / 2\pi$ can therefore be assumed known.

Let us now introduce the generating function of the canonical transformation

$$W = \int_{r^0}^{r} \left[\alpha_1 (I_1, I_2) - \frac{2fM}{\tau} - \frac{I_2^2}{2\pi^2\tau^2} + \frac{fMJ_2R^2}{\tau^3}\right]^{1/2} d\tau + \frac{I_2}{2\pi}\theta =$$
$$\overline{W}(r, \theta, I_1, I_2)$$

which defines the canonical action —angle variables

$$p_1 = \frac{\partial \overline{W}}{\partial r}, \quad p_2 = \frac{\partial \overline{W}}{\partial \theta} = \frac{I_2}{2\pi}, \quad \omega = \frac{\partial \overline{W}}{\partial I_1}, \quad \omega_2 = \frac{\partial \overline{W}}{\partial I_2}$$

Let us turn our attention to a perturbed motion with an analytic perturbation function $H_1 = H_1(I_1, I_2, \omega_1, \omega_2)$, 2π -periodic in ω_i. If the Hessian H_0 is not identically equal to zero, then according to the Kolmogorov theorem the tori $I_1 = \text{const}$ and $I_2 = \text{const}$ are not appreciably deformed provided that the quantity $|H_1|$ is sufficiently small and the conditionally periodic perturbed motion differs slightly from the unperturbed motion. In the present case we have

$$\left|\frac{\partial^2 H_0}{\partial I_i \partial I_j}\right| = \left|\frac{\partial^2 \alpha_1}{\partial I_i \partial I_j}\right| = \left|\frac{\partial (\nu_1, \nu_2)}{\partial (I_1, I_2)}\right| \neq 0$$

where $\nu_1 = \partial\alpha_1 / \partial I_1$, $\nu_2 = \partial\alpha_1 / \partial I_2$ are known functions of the variables I_1 and I_2. Obviously, in the general case they are rationally incommensurable.

Since

$$\frac{\partial^2 \alpha_1}{\partial I_1^2} = -\frac{\partial^2 I_1 / \partial \alpha_1^2}{(\partial I_1 / \partial \alpha_1)^3}$$

where $\partial^2 I_2 / \partial \alpha_1^2 \neq 0$ and $\partial I_1 / \partial \alpha_1 \neq 0$, and hence $\partial \nu_1 / \partial I_1 \neq 0$, the condition $H_0(I_1, I_2) \not\equiv 0$ holds in the region in question. Then, provided that the perturbing function εH_1 is sufficiently small in modulo, we can assert, in accordance with the Kolmogorov theorem on preserving the conditionally periodic motions in Hamiltonian systems, that when the Hamiltonian undergoes a small change [1], then the perturbed motion of the satellite of an axisymmetric planet will be conditionally periodic.

The applicability of the theoretical conclusions obtained depends on the magnitude of the small parameter.

Obviously, in the case of an axisymmetric planet a sufficient condition for preserving the conditionally periodic motions of the satellite can be formulated. This case (see (2)) has the corresponding potential of two fixed centers and the potential of the Barrar's problem [3]. In other cases additional estimates are needed.

REFERENCES

1. K o l m o g o r o v, A. N. On preserving the conditionally periodic motions when the Hamiltonian function undergoes small variations. Dokl. Akad. Nauk SSSR, Vol. 98, No. 4, 1954.

2. K h a s a n o v a, M. Kh. Qualitative study of properties of motion of a satellite of a spheroidal planet. PMM Vol. 41, No. 3, 1977.

3. D e m i n, V. G. Motion of an Artificial Satellite in a Noncentral Gravity Field. Moscow, "Nauka", 1968.

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